

DIOPHANTINE m -TUPLES IN FINITE FIELDS AND MODULAR FORMS

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ABSTRACT. For a prime p , a Diophantine m -tuple in \mathbb{F}_p is a set of m nonzero elements of \mathbb{F}_p with the property that the product of any two of its distinct elements is one less than a square.

In this paper, we present formulas for the number $N^{(m)}(p)$ of Diophantine m -tuples in \mathbb{F}_p for $m = 2, 3$ and 4 . Fourier coefficients of certain modular forms appear in the formula for the number of Diophantine quadruples.

We prove that asymptotically $N^{(m)}(p) = \frac{1}{2^{\binom{m}{2}}} \frac{p^m}{m!} + o(p^m)$, and also show that if $p > 2^{2m-2}m^2$, then there is at least one Diophantine m -tuple in \mathbb{F}_p .

1. INTRODUCTION

A Diophantine m -tuple is a set of m positive integers with the property that the product of any two of its distinct elements is one less than a square. If a set of nonzero rationals has the same property, then it is called a rational Diophantine m -tuple. Diophantus of Alexandria found the first example of a rational Diophantine quadruple $\{1/16, 33/16, 17/4, 105/16\}$, while the first Diophantine quadruple in integers was found by Fermat, and it was the set $\{1, 3, 8, 120\}$. It was proved in [3] that an integer Diophantine sextuple does not exist and that there are only finitely many such quintuples. A folklore conjecture is that there does not exist an integer Diophantine quintuple. On the other hand, it was shown in [6] that there are infinitely many rational Diophantine sextuples (for another construction see [5]), and it is not known if there are rational Diophantine septuples. For a short survey on Diophantine m -tuples see [4].

One can study Diophantine m -tuples over any commutative ring with the unity. In this paper, we consider Diophantine m -tuples in finite fields \mathbb{F}_p , where p is a prime. In this setting, it is natural to ask about the number $N^{(m)}(p)$ of Diophantine m -tuples with elements in \mathbb{F}_p (we consider 0 to be a square in \mathbb{F}_p).

Since half of the elements of \mathbb{F}_p^\times are squares, heuristically, one expects that the randomly chosen m -tuple of different elements in \mathbb{F}_p^\times will have Diophantine property with the probability of $\frac{1}{2^{\binom{m}{2}}}$, i.e. we expect $N^{(m)}(p) = \frac{1}{2^{\binom{m}{2}}} \frac{p^m}{m!} + o(p^m)$. We prove this asymptotic formula at the end of Section 6.

The main theorem of the paper gives an exact formula for the number of Diophantine quadruples $N^{(4)}(p)$ given in terms of the Fourier coefficients of the following modular forms.

Let

$$\begin{aligned} f_1(\tau) &= \sum_{n=1}^{\infty} a(n)q^n \in S_2(\Gamma_0(32)), \\ f_2(\tau) &= \sum_{n=1}^{\infty} b(n)q^n \in S_3\left(\Gamma_0(8), \left(\frac{-2}{\bullet}\right)\right), \\ f_3(\tau) &= \sum_{n=1}^{\infty} c(n)q^n \in S_3\left(\Gamma_0(16), \left(\frac{-4}{\bullet}\right)\right), \\ f_4(\tau) &= \sum_{n=1}^{\infty} d(n)q^n \in S_4(\Gamma_0(8)), \\ f_5(\tau) &= \sum_{n=1}^{\infty} e(n)q^n \in S_5\left(\Gamma_0(4), \left(\frac{-4}{\bullet}\right)\right), \end{aligned}$$

be (unique) newforms in the corresponding spaces of modular forms.

Note that all modular forms except $f_4(\tau)$ are CM forms so we have explicit formulas of their Fourier coefficients which are given in Section 4.2.

Theorem 1.1. *Let p be a prime. Denote by $q(p) = e(p) - 6d(p) + 24b(p) - 24c(p)$. Then*

$$N^{(4)}(p) = \begin{cases} \frac{1}{24 \cdot 64} (p^4 - 24p^3 + 206p^2 - 650p + 477 + q(p)), & \text{if } p \equiv 1 \pmod{8}, \\ \frac{1}{24 \cdot 64} (p^4 - 24p^3 + 236p^2 - 1098p + 1761 + q(p)), & \text{if } p \equiv 3 \pmod{8}, \\ \frac{1}{24 \cdot 64} (p^4 - 24p^3 + 206p^2 - 698p + 573 + q(p)), & \text{if } p \equiv 5 \pmod{8}, \\ \frac{1}{24 \cdot 64} (p^4 - 24p^3 + 236p^2 - 1050p + 1761 + q(p)), & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

The Ramanujan-Petersson conjecture (which is a theorem for holomorphic cusp forms) implies that $q(p) = O(p^2)$, so we have $N^{(4)}(p) = \frac{1}{24 \cdot 64} p^4 + O(p^3)$.

In addition to this, using a more elementary approach of character sums, in Propositions 6.2 and 6.3 we derive formulas for the number of Diophantine pairs $N^{(2)}(p)$ and the number of Diophantine triples $N^{(3)}(p)$ in \mathbb{F}_p . Already for $m = 4$ this method becomes too involved.

For a general m it is natural to ask how large p must be so that there is at least one Diophantine m -tuple in \mathbb{F}_p . In Theorem 6.1, we prove that this is the case if $p > 2^{2m-2}m^2$.

The rest of the paper is organized as follows. In Section 2, we show a correspondence between Diophantine quadruples $\{a, b, c, d\}$ and triples (Q_1, Q_2, Q_3) of \mathbb{F}_p -points on the curve $\mathcal{D}_t : (x^2 - 1)(y^2 - 1) = t$, where $t = abcd$. If $t \neq 0, 1$, the curve

\mathcal{D}_t is birationally equivalent to the elliptic curve $E_t : S^2 = T^3 - 2(t-2)T^2 + t^2T$, with the distinguished point $R = [t, 2t]$ of order 4. Hence we identify $\mathcal{D}_t(\mathbb{F}_p)$ with $\tilde{E}_t(\mathbb{F}_p) := E_t(\mathbb{F}_p) - \{\mathcal{O}, R, 2R, 3R\}$. Conversely, the triple (Q_1, Q_2, Q_3) of points on $\tilde{E}_t(\mathbb{F}_p)$ corresponds to a Diophantine quadruple if and only if $x(Q_1 + Q_2 + Q_3 + R)$ is a square, and if for no two Q_i and Q_j , $i \neq j$, we have that $Q_i = \pm Q_j + kR$, where $k \in \{0, 1, 2, 3\}$. We call such triple admissible.

In Section 3, we find a formula for $N^{(4)}(p)$ by counting admissible triples on $\tilde{E}_t(\mathbb{F}_p)$ for each t . The formula can be written (see Propositions 3.2 and 3.3) as a linear combination of sums of the form $\sum_{t \in X(\mathbb{F}_p)} P(t)^k$, where X is one of the modular curves (for definitions see Section 4.1)

$$X_1(4), X_1(8), X(2, 4), X(2, 8), X(4, 8)$$

and $P(t)$ is the number of \mathbb{F}_p -rational points on the fiber above t of the universal elliptic curve over the modular curve X , and $k \in \{0, 1, 2, 3\}$.

In Section 4, using universal elliptic curves over the modular curves introduced above, we define certain compatible families of ℓ -adic Galois representations such that the trace of Frobenius F_p under these representations is essentially equal to the sums above. On the other hand, these representations are isomorphic to the ℓ -adic realisations of the motives associated to the spaces of cusps forms of weight $k+1$ on the corresponding groups, which enables us to express the traces of Frobenius in terms of the coefficients of the Hecke eigenforms in those spaces.

In Section 5, using the methods from previous section in Propositions 5.1-5.6 we calculate the sums from the formula for $N^{(4)}(p)$, and prove Theorem 1.1.

By using character sums (Weil's estimates), in Section 6 we obtain formulas for $N^{(2)}(p)$ and $N^{(3)}(p)$, and prove Theorem 6.1 together with an asymptotic formula for $N^{(m)}(p)$.

2. CORRESPONDENCE

Let $\{a, b, c, d\}$ be a Diophantine quadruple with elements in \mathbb{F}_p , and let

$$\begin{aligned} ab + 1 &= t_{12}^2, & ac + 1 &= t_{13}^2, & ad + 1 &= t_{14}^2, \\ bc + 1 &= t_{23}^2, & bd + 1 &= t_{24}^2, & cd + 1 &= t_{34}^2. \end{aligned}$$

It follows that $(t_{12}, t_{34}, t_{13}, t_{24}, t_{14}, t_{23}, t = abcd) \in \mathbb{F}_p^7$ defines a point on an algebraic variety \mathcal{C} over \mathbb{F}_p defined by the following equations:

$$\begin{aligned} (t_{12}^2 - 1)(t_{34}^2 - 1) &= t \\ (t_{13}^2 - 1)(t_{24}^2 - 1) &= t \\ (t_{14}^2 - 1)(t_{23}^2 - 1) &= t. \end{aligned}$$

Conversely, the points $(\pm t_{12}, \pm t_{34}, \pm t_{13}, \pm t_{24}, \pm t_{14}, \pm t_{23}, t) \in \mathbb{F}_p^7$ on \mathcal{C} determine two Diophantine quadruples $\pm(a, b, c, d)$ (or one if $-(a, b, c, d) = (a, b, c, d)$), provided that

the elements a, b, c and d are \mathbb{F}_p -rational, distinct and non-zero. Note that $a^2 = (t_{12}^2 - 1)(t_{13}^2 - 1)/(t_{23}^2 - 1)$. Also, if only one element of quadruple is \mathbb{F}_p -rational, then all the elements are \mathbb{F}_p -rational.

The projection $(t_{12}, t_{34}, t_{13}, t_{24}, t_{14}, t_{23}, t) \mapsto t$ defines a fibration of \mathcal{C} over the projective line, and the generic fiber is a cube of $\mathcal{D}_t : (x^2 - 1)(y^2 - 1) = t$. Any point on \mathcal{C} corresponds to the three points $Q_1 = (t_{12}, t_{34})$, $Q_2 = (t_{13}, t_{24})$ and $Q_3 = (t_{14}, t_{23})$ on \mathcal{D}_t . The elements of the quadruple corresponding to these three points are distinct if and only if no two of these points can be transformed from one to another by changing signs and switching coordinates (e.g. for the points (t_{12}, t_{34}) , $(-t_{34}, t_{12})$ and (t_{14}, t_{23}) , we have that $a = d$).

The curve \mathcal{D}_t for $t \in \mathbb{F}_p$ and $t \neq 0, 1$ is birationally equivalent to the elliptic curve

$$E_t : S^2 = T^3 - 2(t - 2)T^2 + t^2T.$$

The map is given by $T = 2(x^2 - 1)y + 2x^2 - (2 - t)$, and $S = 2Tx$. The family E_t over the t -line together with $R = [t, 2t]$, the point of order 4, is the universal elliptic curve over the modular curve $X_1(4)$ (we identify \mathbb{P}^1 with $X_1(4)$). It is easy to see that the affine points on the curve \mathcal{D}_t are in 1-1 correspondence with the set $\tilde{E}_t(\mathbb{F}_p) := E_t(\mathbb{F}_p) \setminus \{\mathcal{O}, R, 2R, 3R\}$.

If $Q \in E_t$ is the point that corresponds to the point $(x, y) \in \mathcal{D}_t$, then the points $-Q$ and $Q + R$ correspond to the points $(-x, y)$ and $(y, -x)$. Hence the following lemma follows.

Lemma 2.1. *The triple $(Q_1, Q_2, Q_3) \in \tilde{E}_t(\mathbb{F}_p)^3$ corresponds to the quadruple whose elements are not distinct if and only if there are two points, Q_i and Q_j , such that $Q_i = \pm Q_j + kR$, where $k \in \{0, 1, 2, 3\}$.*

A short calculation shows that for the point $(S, T) \in E_t(\mathbb{F}_p)$ we have

$$x^2 - 1 = \left(\frac{S}{2T}\right)^2 - 1 = T \left(\frac{T - t}{2T}\right)^2 =: f(T).$$

Since

$$a^2 = \frac{f(Q_1)f(Q_2)f(Q_3)}{t} \equiv x(Q_1)x(Q_2)x(Q_3)t \equiv x(Q_1)x(Q_2)x(Q_3)x(R) \pmod{\mathbb{F}_p^{\times 2}}$$

for the rationality of a it is enough to prove that $x(Q_1)x(Q_2)x(Q_3)x(R)$ is a square in \mathbb{F}_p .

Since the point $(0, 0) \in E(\mathbb{F}_p)$ is a point of order 2, the usual 2-descent homomorphism $E(\mathbb{F}_p) \rightarrow \mathbb{F}_p^\times / \mathbb{F}_p^{\times 2}$, which is for non-torsion points defined by $(T, S) \mapsto T$ (note that $(0, 0) \mapsto 1$), implies the following lemma.

Lemma 2.2. *The triple $(Q_1, Q_2, Q_3) \in \tilde{E}_t(\mathbb{F}_p)^3$ corresponds to the Diophantine quadruple whose elements are \mathbb{F}_p -rational, if and only if*

$$x(Q_1 + Q_2 + Q_3 + R) \text{ is a square.}$$

We call a triple $(Q_1, Q_2, Q_3) \in \tilde{E}_t(\mathbb{F}_p)$ *admissible* if it corresponds to a Diophantine quadruple, i.e. if $x(Q_1 + Q_2 + Q_3 + R)$ is a square, and if there are no two points Q_i and Q_j , $i \neq j$, such that $Q_i = \pm Q_j + kR$ for some $k \in \{0, 1, 2, 3\}$.

3. COUNTING ADMISSIBLE TRIPLES

The main idea of this paper is to count the number $N^{(4)}(p)$ of Diophantine quadruples over \mathbb{F}_p , by counting the admissible triples (Q_1, Q_2, Q_3) .

Since one Diophantine quadruple (a, b, c, d) corresponds to multiple admissible triples, we count each triple with certain weight $w(Q_1, Q_2, Q_3)$: if some Q_i or $Q_i \pm R$ has order 2 then the weight is $w = \frac{1}{24}$ (because one t_{ij} will be equal to 0), otherwise it is $w = \frac{1}{25}$. Note that if there are two Q_i and Q_j such that $2Q_i = \pm R$ and $2Q_j = \pm R$, then the corresponding Diophantine quadruple has the property that $-(a, b, c, d) = (a, b, c, d)$ and we count such a triple with weight $w = \frac{1}{25}$ (unless the third element Q_k or $Q_k \pm R$ has order 2, in which case the weight is $w = \frac{1}{24}$). For $t \in \mathbb{F}_p \setminus \{0, 1\}$, denote by

$$W(t) = \frac{1}{24} \sum_{(Q_1, Q_2, Q_3)} w(Q_1, Q_2, Q_3),$$

where the sum is over all admissible triples $(Q_1, Q_2, Q_3) \in E_t(\mathbb{F}_p)^3$. Thus $W(t)$ is equal to the number of Diophantine quadruples (a, b, c, d) with $abcd = t$. Also for the correct count, Diophantine quadruples corresponding to the singular fiber \mathcal{D}_1 will be counted separately, we denote their number by $W(1)$. For $t \in \mathbb{F}_p, t \neq 0, 1$, denote by $P(t) = \#E_t(\mathbb{F}_p)$.

For every $Q \in \tilde{E}_t(\mathbb{F}_p)$, denote by $[Q]$ the set $\{Q + kR, -Q + kR : k \in \{0, 1, 2, 3\}\}$. We call such a set the class of Q . Note that $\#[Q] = 8$, unless $[Q]$ contains a point of order 2 (different than $2R = [0, 0]$) or a point Q' such that $2Q' = \pm R$, in which case $\#[Q] = 4$.

Proposition 3.1. *a) Let $T \in E_t(\mathbb{F}_p)$ be a point of order two, $T \neq 2R$. Then $x(T)$ is a square if and only if $p \equiv 1 \pmod{4}$.*
b) Let $Q \in E_t(\mathbb{F}_p)$ satisfy $2Q = \pm R$, and let $P \in E_t(\overline{\mathbb{F}_p})$ be such that $2P = Q$. Then $x(Q)$ is a square if and only if the subgroup $\langle P \rangle \leq E_t(\mathbb{F}_p)$ generated by P is \mathbb{F}_p -rational.
c) Let $T \in E_t(\mathbb{F}_p)$ satisfy $2T = \mathcal{O}$ ($T \neq 2R$), and let $P \in E_t(\overline{\mathbb{F}_p})$ be such that $2P = T$. Then $x(T)$ is a square if and only if $P^\sigma - P \in \{\mathcal{O}, 2R\}$, for all $\sigma \in \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$.

Proof. a) The x -coordinates of the points of order two satisfy $x(x^2 + (4-2t)x + t^2x) = 0$. In particular, $x(T) = t - 2 \pm 2\sqrt{1-t} = -(\pm\sqrt{1-t} - 1)^2$ is a square if and only if $\left(\frac{-1}{p}\right) = 1$ (since $\left(\frac{1-t}{p}\right) = 1$). Hence the claim follows.

- b) It follows from the explicit two-descent theory (see Theorem 1.1. in Chapter X of [15]) that there is a bilinear pairing

$$b : E_t(\mathbb{F}_p)/2E_t(\mathbb{F}_p) \times E_t[2] \rightarrow \mathbb{F}_p^\times / \mathbb{F}_p^{\times 2}$$

satisfying

$$e_2(P^\sigma - P, 2R) = \delta_K(b(P, 2R))(\sigma) \text{ for every } \sigma \in \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p),$$

where e_2 is the Weil pairing, and δ_K is the connecting homomorphism for the Kummer sequence associated to the group variety $\mathbb{G}_m/\mathbb{F}_p$. Moreover, $b(P, 2R) \equiv x(Q) \pmod{\mathbb{F}_p^{\times 2}}$. In particular, $x(Q)$ is square if and only if $e_2(P^\sigma - P, 2R) = 1$ for all σ , or equivalently if and only if $P^\sigma - P \in \{\mathcal{O}, 2R\}$ for all σ (since $e_2(2R, 2R) = 1$). The claim follows since $4P = \pm R$.

- c) Same as in b), $x(T)$ is square if and only if $e_2(T^\sigma - T, 2R) = 1$ for all σ . Hence, the claim follows. \square

Remark. Note that half of the points in $E_t(\mathbb{F}_p)$ will have a x -coordinate equal to a square.

3.1. R is not divisible by 2 in $\tilde{E}_t(\mathbb{F}_p)$. In case when $x(R) \neq \square$, we can count triples (Q_1, Q_2, Q_3) by first choosing three different classes $([Q_1], [Q_2], [Q_3])$, and then choosing all possible elements from these classes. Half of these triples will be admissible since (for every $P \in \tilde{E}_t(\mathbb{F}_p)$ precisely one of $x(P)$ and $x(P + R)$ is a square). We consider two cases:

- a) $E_t(\mathbb{F}_p)[2] \cong \mathbb{Z}/2\mathbb{Z}$ ($2R$ is the only point of order two and $x(R) \neq \square$)
All the classes have eight elements and weight $w = \frac{1}{25}$. Denote by b the total number of classes. Then $b = \frac{P(t)-4}{8}$ and

$$W(t) = w \cdot b(b-1)(b-2)2^8 = \frac{(P(t)-20)(P(t)-12)(P(t)-4)}{64}.$$

- b) $E_t(\mathbb{F}_p)[2] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $x(R) \neq \square$ ($T \in E_t(\mathbb{F}_p)$ is a point of order two different than $2R$)

The class $[T]$ contains four elements (and triples containing its elements have weight $w = \frac{1}{24}$), while other $b = \frac{P(t)-8}{8}$ classes contain eight elements. Hence,

$$W(t) = \frac{1}{25}b(b-1)(b-2)2^8 + \frac{1}{24}3 \cdot b(b-1)2^7 = \frac{P(t)(P(t)-8)(P(t)-16)}{64}.$$

In the case when $x(R) = \square$, we consider two cases:

- i) $E_t(\mathbb{F}_p)[2] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $p \equiv 1 \pmod{4}$

Proposition 3.1 implies that $x(T) = \square$, hence there are $b_1 = \frac{P(t)-16}{16}$ eight-element classes $[Q_i]$ for which $x(Q_i)$ is a square, and $b_2 = \frac{P(t)}{16}$ eight-element

classes $[Q_i]$ for which $x(Q_i)$ is not a square. Hence,

$$\begin{aligned} W(t) &= \frac{1}{2^5} b_1(b_1 - 1)(b_1 - 2)2^9 + \frac{3}{2^5} b_1 b_2(b_2 - 1)2^9 + \frac{3}{2^4} (b_1(b_1 - 1) + b_2(b_2 - 1))2^8 \\ &= \frac{P(t)(P(t) - 8)(P(t) - 16)}{64}. \end{aligned}$$

Note that in this case $W(t)$ is equal to the $W(t)$ from the b).

ii) $E_t[2](\mathbb{F}_p) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $p \equiv 3 \pmod{4}$

Proposition 3.1 implies that $x(T) \neq \square$, hence there are $b = \frac{P(t)-8}{16}$ classes $[Q_i]$ for which $x(Q_i)$ is a square, and b classes $[Q_i]$ for which $x(Q_i)$ is not a square. Hence

$$\begin{aligned} W(t) &= \frac{1}{2^5} (b(b-1)(b-2)2^9 + 3b(b-1)b \cdot 2^9) + \frac{6}{2^4} (b \cdot b)2^8 \\ &= \frac{1}{64} (P(t) - 8)(P(t)^2 - 16P(t) + 192). \end{aligned}$$

3.2. $2Q = R$ **and** $x(Q) \neq \square$. Since $2Q = R$ for some $Q \in E_t(\mathbb{F}_p)$, the x -coordinates of all the points in any class $[Q_1]$ is either a square or a non-square. We consider two cases:

a) $E_t(\mathbb{F}_p)[2] \cong \mathbb{Z}/2\mathbb{Z}$ ($2R$ is the only point of order two)

The class $[Q]$ contains four points, while the other $b = \frac{P(t)-8}{8}$ classes contain eight points. All triples have weight $w = \frac{1}{2^5}$. There are precisely $\frac{b}{2}$ classes $[Q_1]$ for which $x(Q_1)$ is equal to a square (since $x(Q)$ is not a square, and half of the points in $E_t(\mathbb{F}_p)$ have a x -coordinate which is a square). Hence

$$\begin{aligned} W(t) &= \frac{1}{2^5} \left(\frac{b}{2} \left(\frac{b}{2} - 1 \right) \left(\frac{b}{2} - 2 \right) 2^9 + 3 \left(\frac{b}{2} \right)^2 \left(\frac{b}{2} - 1 \right) 2^9 + 3! \left(\frac{b}{2} \right)^2 2^8 \right) \\ &= \frac{(P(t) - 8)(P(t)^2 - 28P(t) + 288)}{64}. \end{aligned}$$

b) $E_t(\mathbb{F}_p)[2] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ($T \in E_t(\mathbb{F}_p)$ is a point of order two different than $2R$)

There are three classes with four elements: $[Q]$, $[T]$, and $[T + Q]$.

i) $p \equiv 1 \pmod{4}$

In this case Proposition 3.1 implies that $x(T)$ is a square, and $x(Q + T)$ is not a square. There are $b = \frac{P(t)-16}{8}$ classes with eight elements, half of which

have x -coordinate equal to a square. We calculate

$$\begin{aligned}
W(t) &= \frac{1}{2^5} \left(\frac{b}{2} \left(\frac{b}{2} - 1 \right) \left(\frac{b}{2} - 2 \right) 2^9 + 3 \left(\frac{b}{2} \right)^2 \left(\frac{b}{2} - 1 \right) 2^9 \right) \\
&+ \frac{2 \cdot 3}{2^4} \left(\frac{b}{2} \right) \left(\frac{b}{2} - 1 \right) 2^8 + \frac{2}{2^5} \left(3! \left(\frac{b}{2} \right)^2 \right) 2^8 + \frac{3!}{2^4} \frac{b}{2} 2^7 \\
&+ \frac{3!}{2^4} \frac{b}{2} 2^7 + \frac{3!}{2^5} \frac{b}{2} 2^7 + \frac{3!}{2^4} 2^6 \\
&= \frac{P(t)(P(t)^2 - 24P(t) + 224)}{64}.
\end{aligned}$$

ii) $p \equiv 3 \pmod{4}$

In this case Proposition 3.1 implies that $x(T)$ is not a square, and $x(Q + T)$ is a square. There are $b = \frac{P(t)-16}{8}$ classes with eight elements, half of which have x -coordinate equal to a square. We calculate $W(t)$ as in i).

$$\begin{aligned}
W(t) &= \frac{1}{2^5} \left(\frac{b}{2} \left(\frac{b}{2} - 1 \right) \left(\frac{b}{2} - 2 \right) 2^9 + 3 \left(\frac{b}{2} \right)^2 \left(\frac{b}{2} - 1 \right) 2^9 \right) \\
&+ \frac{3!}{2^4} \left(\frac{b}{2} \right)^2 2^8 + \frac{1}{2^5} \left(3! \left(\frac{b}{2} \right)^2 + 3 \cdot 2 \cdot \frac{b}{2} \left(\frac{b}{2} - 1 \right) \right) 2^8 + \frac{3!}{2^4} \frac{b}{2} 2^7 \\
&+ \frac{3!}{2^4} \frac{b}{2} 2^7 + \frac{3!}{2^5} \frac{b}{2} 2^7 + \frac{3!}{2^4} 2^6 \\
&= \frac{P(t)^3 - 24P(t)^2 + 416P(t) - 3072}{64}.
\end{aligned}$$

3.3. $2Q = R$ **and** $x(Q) = \square$. We consider two cases.

a) $E_t(\mathbb{F}_p) \cong \mathbb{Z}/2\mathbb{Z}$ ($2R$ is the only point of order two)

The class $[Q]$ contains four points, while the other $b = \frac{P(t)-8}{8}$ classes contain eight points. All triples have weight $w = \frac{1}{2^5}$. There are $b_1 = \frac{b-1}{2}$ classes $[Q_1]$ with $x(Q_1) = \square$, and $b_2 = \frac{b+1}{2}$ classes $[Q_1]$ with $x(Q_1) \neq \square$. We have

$$\begin{aligned}
W(t) &= \frac{1}{2^5} (b_1(b_1 - 1)(b_1 - 2) + 3b_1b_2(b_2 - 1)) 2^9 + \frac{1}{2^5} (3b_1(b_1 - 1) + 3b_2(b_2 - 1)) 2^8 \\
&= \frac{(P(t) - 16)(P(t)^2 - 20P(t) + 192)}{64}.
\end{aligned}$$

b) $E_t(\mathbb{F}_p) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ($T \in E_t(\mathbb{F}_p)$ is a point of order two different than $2R$)

There are three classes with four elements: $[Q]$, $[T]$, and $[T + Q]$, and $b = \frac{P(t)-16}{8}$ classes with eight elements.

i) $p \equiv 1 \pmod{4}$

Proposition 3.1 implies that $x(T)$ and $x(T + Q)$ are both squares. Then

$$b_1 = \frac{b-2}{2} \text{ and } b_2 = \frac{b+2}{2}.$$

$$\begin{aligned} W(t) &= \frac{1}{2^5} (b_1(b_1-1)(b_1-2) + 3b_1b_2(b_2-1)) 2^9 + (2+2) \frac{1}{2^5} (3b_1(b_1-1) + 3b_2(b_2-1)) 2^8 \\ &+ \frac{2}{2^4} 3!b_1 2^7 + \frac{1}{2^5} 3!b_1 2^7 + \frac{1}{2^4} 3!2^6 \\ &= \frac{(P(t)-16)(P(t)^2-8P(t)+96)}{64}. \end{aligned}$$

ii) $p \equiv 3 \pmod{4}$

Proposition 3.1 implies that $x(T)$ and $x(T+Q)$ are not squares. Then $b_1 = b_2 = \frac{b}{2}$, and we calculate

$$\begin{aligned} W(t) &= \frac{1}{2^5} (b_1(b_1-1)(b_1-2) + 3b_1b_2(b_2-1)) 2^9 + \frac{1}{2^5} (3b_1(b_1-1) + 3b_2(b_2-1)) 2^8 \\ &+ (1+2) \frac{1}{2^5} (3!b_1b_2) 2^8 + \frac{1}{2^4} 3!b_1 2^7 + \frac{1}{2^4} 3!b_2 2^7 + \frac{1}{2^5} 3!b_1 2^7 + \frac{1}{2^4} 3!2^6 \\ &= \frac{P(t)^3 - 24P(t)^2 + 416P(t) - 3072}{64}. \end{aligned}$$

3.4. Putting everything together. For the fixed prime p we define the following sets:

$$\begin{aligned} T_0 &= \{t \in \mathbb{F}_p^\times / \{1\} : E_t[2](\mathbb{F}_p) = \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}} \text{ and } x(R) = \square\} \\ T_1 &= \{t \in \mathbb{F}_p^\times / \{1\} : E_t[2](\mathbb{F}_p) = \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}\} \\ T_2 &= \{t \in \mathbb{F}_p^\times / \{1\} : E_t[2](\mathbb{F}_p) = \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}} \text{ and } 2Q = R \text{ for some } Q \in E_t(\mathbb{F}_p)\} \\ T_3 &= \{t \in \mathbb{F}_p^\times / \{1\} : 2Q = R \text{ for some } Q \in E_t(\mathbb{F}_p)\} \\ T_4 &= \{t \in T_3 : \langle S \rangle \text{ is } \mathbb{F}_p\text{-rational, where } 2S \in E_t(\mathbb{F}_p) \text{ and } 4S = R \text{ for some } S \in E_t\} \\ T_5 &= \{t \in T_4 : E_t[2](\mathbb{F}_p) = \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}\}. \end{aligned}$$

Remark. Note that it follows from Proposition 3.1 that

$$T_4 = \{t \in \mathbb{F}_p^\times / \{1\} : 2Q = R \text{ and } x(Q) = \square \text{ for some } Q \in E_t(\mathbb{F}_p)\}.$$

Also, if $p \equiv 3 \pmod{4}$, then $T_2 = T_5$.

We have the following proposition.

Proposition 3.2. *a) If $p \equiv 1 \pmod{4}$, then*

$$\begin{aligned} 24 \sum_{t \neq 0,1} W(t) &= \sum_{t \neq 0,1} \left(\frac{1}{64} P(t)^3 - \frac{9}{16} P(t)^2 + \frac{23}{4} P(t) - 15 \right) \\ &+ \sum_{t \in T_1} \left(\frac{3}{16} P(t)^2 - \frac{15}{4} P(t) + 15 \right) - \sum_{t \in T_2} \left(\frac{3}{4} P(t) - 21 \right) \\ &+ \sum_{t \in T_3} \left(\frac{9}{4} P(t) - 21 \right) - 12 \sum_{t \in T_4} 1 - 12 \sum_{t \in T_5} 1. \end{aligned}$$

b) If $p \equiv 3 \pmod{4}$ then

$$\begin{aligned} 24 \sum_{t \neq 0,1} W(t) &= \sum_{t \neq 0,1} \left(\frac{1}{64} P(t)^3 - \frac{9}{16} P(t)^2 + \frac{23}{4} P(t) - 15 \right) \\ &+ \sum_{t \in T_1} \left(\frac{3}{16} P(t)^2 - \frac{15}{4} P(t) + 15 \right) - \sum_{t \in T_2} \left(\frac{3}{4} P(t) + 3 \right) \\ &+ \sum_{t \in T_3} \left(\frac{9}{4} P(t) - 21 \right) + \sum_{t \in T_0} (3P(t) - 24) - 12 \sum_{t \in T_4} 1 + 12 \sum_{t \in T_5} 1. \end{aligned}$$

3.5. Calculating $W(1)$. The curve $\mathcal{D}_1 : (x^2 - 1)(y^2 - 1) = 1$ is birationally equivalent to the genus zero curve $E_1 : S^2 = T(T + 1)^2$. Analysis similar (but easier) to the one in Section 2 yields the following proposition.

Proposition 3.3.

$$24 \cdot W(1) = \begin{cases} \frac{(p-9)(p^2-18p+113)}{32}, & \text{if } p \equiv 1 \pmod{8}, \\ \frac{(p-3)(p-11)(p-19)}{32}, & \text{if } p \equiv 3 \pmod{8}, \\ \frac{(p-5)(p-9)(p-13)}{32}, & \text{if } p \equiv 5 \pmod{8}, \\ \frac{(p-7)(p-11)(p-15)}{32}, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

4. FAMILIES OF UNIVERSAL ELLIPTIC CURVES AND ℓ -ADIC REPRESENTATIONS

4.1. Modular curves and cusps. For $M, N \geq 1$, $M|N$, we denote by $Y(M, N)$ the quotient of the upper half plane by the congruence subgroup $\Gamma_1(N) \cap \Gamma^0(M)$. As a modular curve (irreducible, connected and defined over $\mathbb{Q}(\zeta_M)$) $Y(M, N)$ parametrizes elliptic curves E together with the points P and Q of order M and N , such that P and Q generate subgroup of order MN and the Weil pairing $e_{N/M}$ between the points P and $\frac{N}{M}Q$ is equal to the fixed primitive M -th root of unity, i.e. $e_{N/M}(P, \frac{N}{M}Q) = e^{2\pi i/M}$. We denote by $X(M, N)$ the compactification of $Y(M, N)$. For more information on modular curves $Y(M, N)$ see [8].

In Section 5, we will need to know the number of \mathbb{F}_p -rational cusps on modular curves $X_1(8)_{\mathbb{F}_p}$, $X(2, 4)_{\mathbb{F}_p}$, $X(2, 8)_{\mathbb{F}_p}$ and $X(4, 8)_{\mathbb{F}_p}$. Following [2, Section 2], we briefly explain how to calculate the field of definition of cusps on $X(M, N)$.

Let r be a divisor of N . The cusps of $X(M, N)$ represented by the points $(a : b) \in \mathbb{P}^1(\mathbb{Q})$, where a, b are co-prime integers with $\gcd(b, N) = r$, all have the same field of definition, $\mathbb{Q}(\zeta_M) \leq F_r \leq \mathbb{Q}(\zeta_N)$. If we canonically identify $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ with $(\mathbb{Z}/N\mathbb{Z})^\times$, then F_r is the fixed field of the group H_r acting on $\mathbb{Q}(\zeta_N)$, where $H_r = H_r^0 := \{s \in (\mathbb{Z}/N\mathbb{Z})^\times : s \equiv 1 \pmod{\text{lcm}(M, N/r)}\}$, if $\gcd(Mr, N) > 2$, and $H_r = H_r^0 \cdot \{\pm 1\}$ otherwise.

It follows immediately that all four cusps of $X(2, 4)$ are \mathbb{Q} -rational (i.e. $c(2, 4) = 4$), and that the number $c(2, 8)$ of \mathbb{F}_p -rational cusps of $X(2, 8)_{\mathbb{F}_p}$ is equal to

$$c(2, 8) = \begin{cases} 10, & \text{if } p \equiv 1 \pmod{8}, \\ 4, & \text{if } p \equiv 3 \pmod{8}, \\ 6, & \text{if } p \equiv 5 \pmod{8}, \\ 8, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Moreover, the curve $X(4, 8)_{\mathbb{Q}(i)}$ has eight cusps defined over $\mathbb{Q}(i)$ and eight cusps defined over $\mathbb{Q}(\zeta_8)$, hence the number of \mathbb{F}_p -rational cusps is equal to

$$c(4, 8) = \begin{cases} 16, & \text{if } p \equiv 1 \pmod{8}, \\ 8, & \text{if } p \equiv 5 \pmod{8}. \end{cases}$$

The number of \mathbb{F}_p -rational cusps on modular curve $X_1(8)_{\mathbb{F}_p}$ is equal to

$$c(8) = \begin{cases} 6, & \text{if } p \equiv 1 \pmod{8}, \\ 4, & \text{if } p \equiv 3 \pmod{8}, \\ 4, & \text{if } p \equiv 5 \pmod{8}, \\ 6, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

The curves $X_1(4)$, $X(2, 4)$, $X_1(8)$ and $X(2, 8)$ have genus zero, while the curve $X(4, 8)$ has genus one.

4.2. Modular forms. Here we collect some facts about the spaces of modular forms related to the modular curves from the previous subsection. They can be checked using Sage [13] and LMFDB database [9].

Proposition 4.1. *Denote by T_p the p -th Hecke operator acting on the space of cusp forms $S_3(\Gamma_1(8) \cap \Gamma^0(2))$. We have*

- a) $\dim S_3(\Gamma_1(4)) = \dim S_4(\Gamma_1(4)) = 0$ and $S_5(\Gamma_1(4)) = \mathbb{C} \cdot f_5(\tau)$,
- b) $\dim S_3(\Gamma_1(8) \cap \Gamma^0(2)) = 3$ and $\text{Trace}(T_p) = 2b(p) + c(p)$,
- c) $S_3(\Gamma_1(8)) = \mathbb{C} \cdot f_2(\tau)$,
- d) $S_3(\Gamma_1(4) \cap \Gamma^0(2)) = 0$ and $S_4(\Gamma_1(4) \cap \Gamma^0(2)) = \mathbb{C} \cdot f_4(\tau)$.

Modular forms $f_1(\tau)$, $f_2(\tau)$, $f_3(\tau)$ and $f_5(\tau)$ are CM forms, and their Fourier coefficients are given in the following proposition. For some standard facts about CM modular forms see [12, p.9].

Proposition 4.2. *Let p be an odd prime and $q = e^{2\pi i\tau}$. We have*

a) $f_1(\tau) = \eta^2(4\tau)\eta^2(8\tau) = q \prod_{n=1}^{\infty} (1 - q^{4n})^2 (1 - q^{8n})^2$, and

$$a(p) = \begin{cases} \pm 2x, & \text{if } p \equiv 1 \pmod{4} \text{ and } p = x^2 + 4y^2 \\ 0, & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

b) $f_2(\tau) = \eta^2(\tau)\eta(2\tau)\eta(4\tau)\eta^2(8\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{2n})(1 - q^{4n})(1 - q^{8n})^2$,
and

$$b(p) = \begin{cases} 2(x^2 - 2y^2), & \text{if } p \equiv 1, 3 \pmod{8} \text{ and } p = x^2 + 2y^2 \\ 0, & \text{if } p \equiv 5, 7 \pmod{8}, \end{cases}$$

c) $f_3(\tau) = \eta^6(4\tau) = q \prod_{n=1}^{\infty} (1 - q^{4n})^6$, and

$$c(p) = \begin{cases} \pm 2(x^2 - 4y^2), & \text{if } p \equiv 1 \pmod{4} \text{ and } p = x^2 + 4y^2 \\ 0, & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

d) $f_4(\tau) = \eta^4(2\tau)\eta^4(4\tau) = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4$,

e) $f_5(\tau) = \eta^4(\tau)\eta^2(2\tau)\eta^4(4\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^4 (1 - q^{2n})^2 (1 - q^{4n})^4$, and

$$e(p) = \begin{cases} 2p^2 - 16x^2y^2, & \text{if } p \equiv 1 \pmod{4} \text{ and } p = x^2 + y^2 \\ 0, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

4.3. Families of universal elliptic curves. Let E^1, E^2, E^3 and E^4 be elliptic surfaces fibered over the modular curves $X_1(4)$, $X(2, 4)$, $X_1(8)$ and $X(2, 8)$ defined by affine equations (given with the sections of the corresponding orders):

$$E^1 : y^2 = X(X^2 - 2(t_1 - 2)X + t_1^2), \quad P_1 = [t_1, 2t_1]; 4P_1 = \mathcal{O}$$

$$E^2 : y^2 = X(X + t_2^2 - 2t_2 + 1)(X + t_2^2 + 2t_2 + 1),$$

$$P_2 = [1 - t_2^2, 2(1 - t_2^2)], \quad T_2 = [-t_2^2 + 2t_2 - 1, 0]; 4P_2 = 2T_2 = \mathcal{O}$$

$$E^3 : y^2 = X(X^2 - 2(t_3^4 - 2t_3^2 - 1) + (t_3 - 1)^4(t_3 + 1)^4),$$

$$Q_3 = [(t_3 - 1)(t_3 + 1)^3, 2t_3(t_3 - 1)(t_3 + 1)^3]; 8Q_3 = \mathcal{O}$$

$$E^4 : y^2 = X \left(X + \frac{64t_4^4}{(t_4^2 + 1)^4} \right) \left(X + \frac{4(t_4 - 1)^4(t_4 + 1)^4}{(t_4^2 + 1)^4} \right),$$

$$Q_4 = \left[\frac{-16t_4(t_4 - 1)(t_4 + 1)^3}{(t_4^2 + 1)^4}, \frac{32t_4(t_4 - 1)(t_4 + 1)^3(t_4^2 - 2t_4 - 1)}{(t_4^2 + 1)^5} \right], T_4 = \left[-\frac{64t_4^4}{(t_4^2 + 1)^4}, 0 \right];$$

$$8Q_4 = 2T_4 = \mathcal{O}$$

$$E^5 : y^2 = X(X + t_5^4 - 2t_5^2 + 1)(X + t_5^4 + 2t_5^2 + 1),$$

$$Q_5 = [1 - t_5^4, 2 - 2t_5^4], \quad Q'_5 = [-t_5^4 + 2it_5^3 + 2t_5^2 - 2it_5 - 1, 2t_5^5 - 4it_5^4 - 4t_5^3 + 4it_5^2 + 2t_5];$$

$$4Q_5 = 4Q'_5 = \mathcal{O}.$$

together with the maps

$$\begin{aligned} h_1 : E^1 &\rightarrow X_1(4) & (X, Y, t_1) &\mapsto t_1, \\ h_2 : E^2 &\rightarrow X(2, 4) & (X, Y, t_2) &\mapsto t_2, \\ h_3 : E^3 &\rightarrow X_1(8) & (X, Y, t_3) &\mapsto t_3, \\ h_4 : E^4 &\rightarrow X(2, 8) & (X, Y, t_4) &\mapsto t_4, \\ h_5 : E^4 &\rightarrow X(4) & (X, Y, t_5) &\mapsto t_4. \end{aligned}$$

Here we identify modular curves $X_1(4)$, $X(2, 4)$, $X_1(8)$, $X(2, 8)$ and $X(5)$ with \mathbb{P}^1 using parameters t_1 , t_2 , t_3 , t_4 and t_5 .

We have the natural maps

$$\begin{aligned} g_1 : X(2, 4) &\rightarrow X_1(4), & (E, T_2, P_2) &\mapsto (E, P_2), & t_1 &= 1 - t_2^2 \\ g_2 : X_1(8) &\rightarrow X_1(4), & (E, Q_3) &\mapsto (E, 2Q_3), & t_1 &= (t_3^2 - 1)^2 \\ g_3 : X(2, 8) &\rightarrow X_1(4), & (E, T_4, Q_4) &\mapsto (E, 2Q_4), & t_1 &= \frac{16t_4^2(t_4 - 1)^2(t_4 + 1)^2}{(t_4^2 + 1)^4}, \\ g_4 : X(4) &\rightarrow X_1(4), & (E, Q_5, Q_5') &\mapsto (E, Q_5), & t_1 &= 1 - t_5^4. \end{aligned}$$

Elliptic surfaces E^1, E^2, E^3 and E^4 are universal elliptic curves over the modular curves $X_1(4), X(2, 4), X_1(8), X(2, 8)$ and $X(4)$ respectively. Note that $X(4)$ is defined over $\mathbb{Q}(i)$.

4.4. Model for $X(4, 8)$. In this section we calculate $\#X(4, 8)(\mathbb{F}_p)$, where $p \equiv 1 \pmod{4}$. Denote by t'_5 and t'_4 pullbacks of function t_4 and t_5 on $X(2, 8)$ and $X(4)$ along the natural maps $X(4, 8) \rightarrow X(2, 8)$, $(E, P, Q) \mapsto (E, 2P, Q)$ and $X(4, 8) \rightarrow X(4)$, $(E, P, Q) \mapsto (E, P, 2Q)$. Then we have

$$1 - t_5'^4 = \frac{16t_4'^2(t_4' - 1)^2(t_4' + 1)^2}{(t_4'^2 + 1)^4},$$

which implies

$$(t_5'(t_4'^2 + 1))^2 = \pm(t_4'^2 - 2t_4' - 1)(t_4'^2 + 2t_4' - 1).$$

The two genus one curves $y^2 = \pm(x^2 - 2x - 1)(x^2 + 2x - 1)$ are isomorphic to the conductor 32 elliptic curve $C : y^2 = x^3 - x$, hence, over $\mathbb{Q}(i)$ the modular curve $X(4, 8)$ is isomorphic to C (since $X(4, 8)$ is connected). For a different proof see Lemma 13 in [11].

Therefore, for prime $p \equiv 1 \pmod{4}$ (which splits in $\mathbb{Q}(i)$) we have that

$$\#X(4, 8)(\mathbb{F}_p) = \#C(\mathbb{F}_p) = p + 1 - a(p),$$

since the modular form $f_1(\tau) = \sum_{n=1}^{\infty} a(n)q^n$ corresponds to C by the modularity theorem (as it is the only newform in $S_2(\Gamma_0(32))$).

4.5. Compatible families of ℓ -adic Galois representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. To each of these elliptic surfaces and to an integer k , we can associate two compatible families of ℓ -adic Galois representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. To ease notation, we denote by Γ_j , for $j = 1, 2, 3, 4$, groups $\Gamma_1(4), \Gamma(2, 4), \Gamma_1(8)$ and $\Gamma(2, 8)$ respectively, and by $X(\Gamma_j)$ the corresponding modular curve. Let $k \geq 0$ be an integer and $j \in \{1, 2, 3, 4\}$.

We define the representation $\rho_{j,\ell}^k$ of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as follows. Let $X(\Gamma_j)^0$ be the complement in $X(\Gamma_j)$ of the cusps. Denote by i the inclusion of $X(\Gamma_j)^0$ into $X(\Gamma_j)$, and by $h_j' : E^j \rightarrow X(\Gamma_j)^0$ the restriction of h_j . For a prime ℓ we obtain a sheaf

$$\mathcal{F}_\ell^j = R^1 h_{j*}' \mathbb{Q}_\ell$$

on $X(\Gamma_j)^0$, and also a sheaf $i_* \text{Sym}^k \mathcal{F}_\ell$ on $X(\Gamma_j)$ (here \mathbb{Q}_ℓ is the constant sheaf on the elliptic surface E_j , and R^1 is derived functor). The action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the \mathbb{Q}_ℓ -space

$$W_\ell^j = H_{et}^1(X(\Gamma_j) \otimes \overline{\mathbb{Q}}, i_* \text{Sym}^k \mathcal{F}_\ell^j)$$

defines an ℓ -adic representation $\rho_{j,\ell}^k$.

The second family, $\tilde{\rho}_{j,\ell}^k$, is ℓ -adic realization of the motive associated to the spaces of cusp forms $S_{k+1}(\Gamma_j)$. For the construction see [14, Section 5].

Similarly as in [1, Section 3], since elliptic surface E_j is the universal elliptic curve over the modular curve $X(\Gamma_j)$, we can argue that these two representations are isomorphic, i.e. $\rho_{j,\ell}^k \sim \tilde{\rho}_{j,\ell}^k$. In particular, we will frequently use the following proposition.

Proposition 4.3. *Let $k \geq 0$ be an integer and $j \in \{1, 2, 3, 4\}$. Denote by B the set of Hecke eigenforms in $S_{k+2}(\Gamma_j)$. If $p \nmid 2\ell$ is a prime, then*

$$\text{Trace}(\rho_{j,\ell}^k(\text{Frob}_p)) = \sum_{f \in B} a_f(p),$$

where $a_f(p)$ is the p -th Fourier coefficient of the eigenform f .

4.6. Traces of Frobenius. To simplify notation, denote $\mathcal{F} = R^1 h_{j*}' \mathbb{Q}_\ell$ and $W = H_{et}^1(X(\Gamma_j) \otimes \overline{\mathbb{Q}}, i_* \mathcal{F})$. The Lefschetz fixed point formula and standard facts about elliptic curves and elliptic surfaces over finite fields give the following theorem.

Theorem 4.4. *The following is true:*

(1)

$$\text{Tr}(\text{Frob}_p|W) = - \sum_{t \in X(\Gamma_j)(\mathbb{F}_p)} \text{Tr}(\text{Frob}_p|(i_* \mathcal{F})_t).$$

(2) *If the fiber $E_t^j := h_j^{-1}(t)$ is smooth, then*

$$\text{Tr}(\text{Frob}_p|(i_* \mathcal{F})_t) = \text{Tr}(\text{Frob}_p|H^1(E_t^j, \mathbb{Q}_\ell)) = p + 1 - \#E_t^j(\mathbb{F}_p).$$

Furthermore,

$$\text{Tr}(\text{Frob}_p|(i_* \text{Sym}^2 \mathcal{F})_t) = \text{Tr}(\text{Frob}_p|(i_* \mathcal{F})_t)^2 - p,$$

and

$$\mathrm{Tr}(\mathrm{Frob}_p|(i_*\mathrm{Sym}^3\mathcal{F})_t) = \mathrm{Tr}(\mathrm{Frob}_p|(i_*\mathcal{F})_t)^3 - 2p \cdot \mathrm{Tr}(\mathrm{Frob}_p|(i_*\mathcal{F})_t).$$

(3) If the fiber E_t^j is singular, then

$$\mathrm{Tr}(\mathrm{Frob}_p|(i_*\mathcal{F})_t) = \begin{cases} 1 & \text{if the fiber is split multiplicative,} \\ -1 & \text{if the fiber is nonsplit multiplicative,} \\ 0 & \text{if the fiber is additive.} \end{cases}$$

Furthermore,

$$\mathrm{Tr}(\mathrm{Frob}_p|(i_*\mathrm{Sym}^2\mathcal{F})_t) = \mathrm{Tr}(\mathrm{Frob}_p|(i_*\mathcal{F})_t)^2,$$

and

$$\mathrm{Tr}(\mathrm{Frob}_p|(i_*\mathrm{Sym}^3\mathcal{F})_t) = \mathrm{Tr}(\mathrm{Frob}_p|(i_*\mathcal{F})_t)^3.$$

5. RESULTS

5.1. $X_1(4)$. The universal elliptic curve E^1 over $X_1(4)$ has three singular fibers (over the cusps): additive $t = \infty$, split multiplicative $t = 0$, and fiber $t = 1$ which is split multiplicative if $p \equiv 1 \pmod{4}$ and nonsplit multiplicative if $p \equiv 3 \pmod{4}$. Denote by $\mathcal{F} = R^1h'_{1*}\mathbb{Q}_\ell$.

Proposition 5.1. a)

a)

$$\sum_{t \neq 0,1} P(t) = \begin{cases} p^2 - p & \text{if } p \equiv 1 \pmod{4}, \\ p^2 - p - 2 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

b)

$$\sum_{t \neq 0,1} P(t)^2 = \begin{cases} p^3 + p^2 - p - 1, & \text{if } p \equiv 1 \pmod{4}, \\ p^3 + p^2 - 5p - 5, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

c)

$$\sum_{t \neq 0,1} P(t)^3 = \begin{cases} p^4 + 4p^3 - 4p - 3 + e(p), & \text{if } p \equiv 1 \pmod{4}, \\ p^4 + 4p^3 - 6p^2 - 20p - 11 + e(p), & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. a) Theorem 4.4 implies that

$$\begin{aligned} \mathrm{Tr}(\mathrm{Frob}_p|W) &= - \sum_{t \in \text{cusps}} \mathrm{Tr}(\mathrm{Frob}_p|(i_*\mathcal{F})_t) - \sum_{t \neq 0,1} (p + 1 - P(t)), \\ &= \sum_{t \neq 0,1} P(t) - (p^2 - p - 2) - \sum_{t \in \text{cusps}} \mathrm{Tr}(\mathrm{Frob}_p|(i_*\mathcal{F})_t), \end{aligned}$$

and the claim follows since $\dim(S_3(\Gamma_1(4))) = 0$ so representation W is zero.

b) Since the trace at every singular fiber is 1, Theorem 4.4 implies that

$$\begin{aligned} \text{Tr}(Frob_p|W) &= - \sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_* \text{Sym}^2 \mathcal{F})_t) - \sum_{t \neq 0,1} ((p+1-P(t))^2 - p), \\ &= - \sum_{t \neq 0,1} P(t)^2 - p^3 + p^2 + p + 2 + 2(p+1) \sum_{t \neq 0,1} P(t) - 3. \end{aligned}$$

The claim follows from the part a) since $\dim(S_4(\Gamma_1(4))) = 0$, hence $\text{Tr}(Frob_p|W) = 0$.

c) Theorem 4.4 implies that

$$\begin{aligned} \text{Tr}(Frob_p|W) &= - \sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_* \text{Sym}^3 \mathcal{F})_t) - \sum_{t \neq 0,1} ((p+1-P(t))^3 - 2p(p+1-P(t))), \\ &= \sum_{t \neq 0,1} P(t)^3 - 3(p+1) \sum_{t \neq 0,1} P(t)^2 + (3(p+1)^2 - 2p) \sum_{t \neq 0,1} P(t) \\ &\quad - (p-2)(p+1)^3 + 2p(p-2)(p+1) - \sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_* \text{Sym}^3 \mathcal{F})_t). \end{aligned}$$

The claim follows from the parts a) and b) and Proposition 4.1a) (hence $\text{Tr}(Frob_p|W) = e(p)$). \square

5.2. $X(2, 8)$. Universal elliptic curve E^4 over $X(2, 8)$ has 10 singular fibers: $t_4 = \pm i$ (two cusps above $t_1 = \infty$) and $t_4^2 + 2t_4 - 1 = 0$ and $t_4^2 - 2t_4 - 1 = 0$ (four cusps above $t_1 = 0$) which are split multiplicative if $p \equiv 1 \pmod{4}$ and nonsplit multiplicative otherwise, and split multiplicative $t_4 = \pm 1, 0, \infty$ (four cusps above $t_1 = 0$). Denote $\mathcal{F} = R^1 h_{4*} \mathbb{Q}_\ell$.

Proposition 5.2. *a)*

$$\sum_{t \in T_2} 1 = \begin{cases} \frac{p-9}{8}, & \text{if } p \equiv 1 \pmod{8}, \\ \frac{p-3}{8}, & \text{if } p \equiv 3 \pmod{8}, \\ \frac{p-5}{8}, & \text{if } p \equiv 5 \pmod{8}, \\ \frac{p-7}{8}, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

b)

$$\sum_{t \in T_2} P(t) = \begin{cases} \frac{p^2-8p+1+2b(p)+c(p)}{8}, & \text{if } p \equiv 1 \pmod{8}, \\ \frac{p^2-2p+1+2b(p)}{8}, & \text{if } p \equiv 3 \pmod{8}, \\ \frac{p^2-4p+1+c(p)}{8}, & \text{if } p \equiv 5 \pmod{8}, \\ \frac{p^2-6p-7}{8}, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Proof. a) Since T_2 is equal to the image of \mathbb{F}_p -points (which are not cusps) on $X(2, 8)_{\mathbb{F}_p}$ under the natural map $g_3 : X(2, 8) \rightarrow X_1(4)$ of degree 8, we have $\sum_{t \in T_2} 1 = \frac{p+1-c(2,8)}{8}$,

since $\#X(2, 8)(\mathbb{F}_p) = p + 1$. The claim follows.

b) From the definition of T_2 we have $\sum_{t \in T_2} P(t) = \frac{1}{8} \sum_{\substack{t \in g_3(X(2, 8)(\mathbb{F}_p)) \\ t \notin \text{cusps}}} P(t)$. Theorem 4.4

implies that (the sum is over $X(2, 8)(\mathbb{F}_p)$)

$$\begin{aligned} \text{Tr}(Frob_p|W) &= - \sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_*\mathcal{F})_t) - \sum_{t \notin \text{cusps}} (p + 1 - P(t)), \\ &= - \sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_*\mathcal{F})_t) - (p + 1)(p + 1 - c(2, 8)) + \sum_{t \notin \text{cusps}} P(t). \end{aligned}$$

It follows from Proposition 4.1b) that $\text{Tr}(Frob_p|W) = 2b(p) + c(p)$. The claim follows since Theorem 4.4 implies

$$\sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_*\mathcal{F})_t) = \begin{cases} 10, & \text{if } p \equiv 1 \pmod{8}, \\ 4, & \text{if } p \equiv 3 \pmod{8}, \\ 6, & \text{if } p \equiv 5 \pmod{8}, \\ 0, & \text{if } p \equiv 7 \pmod{8}, \end{cases}$$

□

and since $b(p) = 0$ if $p \equiv 5, 7 \pmod{8}$ and $c(p) = 0$ if $p \equiv 3, 7 \pmod{8}$.

5.3. $X_1(8)$. Universal elliptic curve E^3 over $X_1(8)$ has 6 singular fibers: split multiplicative $t_3 = \infty$ and $t_3 = \pm 1$ (two cusps above $t_1 = 0$) and $t_3 = 0, \pm\sqrt{2}$ (three cusps above $t_1 = 1$) which are split multiplicative if $p \equiv 1 \pmod{4}$ and nonsplit multiplicative otherwise. Denote $\mathcal{F} = R^1h'_{3*}\mathbb{Q}_\ell$.

Proposition 5.3. a)

$$\sum_{t \in T_3} 1 = \begin{cases} \frac{3p-11}{8}, & \text{if } p \equiv 1 \pmod{8}, \\ \frac{3p-9}{8}, & \text{if } p \equiv 3 \pmod{8}, \\ \frac{3p-7}{8}, & \text{if } p \equiv 5 \pmod{8}, \\ \frac{3p-13}{8}, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

b)

$$\sum_{t \in T_3} P(t) = \begin{cases} \frac{3p^2-8p+2b(p)-c(p)+3}{8}, & \text{if } p \equiv 1 \pmod{8}, \\ \frac{3p^2-6p+2b(p)-5}{8}, & \text{if } p \equiv 3 \pmod{8}, \\ \frac{3p^2-4p-c(p)+3}{8}, & \text{if } p \equiv 5 \pmod{8}, \\ \frac{3p^2-10p-13}{8}, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Proof. a) By definition, T_3 is equal to the image of \mathbb{F}_p -points (which are not cusps) on $X_1(8)$ under the natural map $g_2 : X_1(8) \rightarrow X_1(4)$ of degree 4. Since we have

$p + 1 - c(8) = 4 \sum_{t \in T_2} 1 + 2 \sum_{t \in T_3} 1$, the claim follows.

b) From the definition of T_3 it follows that

$$4 \sum_{t \in T_2} P(t) + 2 \sum_{t \in T_3 - T_2} P(t) = \sum_{\substack{t \in g_2(X_1(8)(\mathbb{F}_p)) \\ t \notin \text{cusps}}} P(t),$$

hence $\sum_{t \in T_3} P(t) = \frac{1}{2} \sum_{\substack{t \in g_2(X_1(8)(\mathbb{F}_p)) \\ t \notin \text{cusps}}} P(t) - \sum_{t \in T_2} P(t)$. Theorem 4.4 implies that

$$\begin{aligned} \text{Tr}(Frob_p|W) &= - \sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_*\mathcal{F})_t) - \sum_{t \notin \text{cusps}} (p + 1 - P(t)), \\ &= - \sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_*\mathcal{F})_t) - (p + 1)(p + 1 - c(8)) + \sum_{t \notin \text{cusps}} P(t), \end{aligned}$$

where the sums are over $X_1(8)(\mathbb{F}_p)$. It follows from Proposition 4.1c) that $\text{Tr}(Frob_p|W) = b(p)$, and the claim follows. Note that Theorem 4.4 implies

$$\sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_*\mathcal{F})_t) = \begin{cases} 6, & \text{if } p \equiv 1 \pmod{8}, \\ 2, & \text{if } p \equiv 3 \pmod{8}, \\ 4, & \text{if } p \equiv 5 \pmod{8}, \\ 0, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

□

5.4. $X(2, 4)$. Universal elliptic curve E^2 over $X(2, 4)$ has 4 singular fibers: $t_2 = 0$ (the cusp above $t_1 = 1$) and $t_2 = \infty$ which are split multiplicative if $p \equiv 1 \pmod{4}$ and nonsplit multiplicative otherwise, and split multiplicative $t_2 = \pm 1$ (two cusps above $t_1 = 0$). Denote $\mathcal{F} = R^1 h'_{2*} \mathbb{Q}_\ell$.

Proposition 5.4. a)

$$\sum_{t \in T_1} 1 = \frac{p-3}{2},$$

b)

$$\sum_{t \in T_1} P(t) = \begin{cases} \frac{(p-1)^2}{2}, & \text{if } p \equiv 1 \pmod{4}, \\ \frac{p^2-2p-3}{2}, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

c)

$$\sum_{t \in T_1} P(t)^2 = \begin{cases} \frac{p^3+1-d(p)}{2}, & \text{if } p \equiv 1 \pmod{4}, \\ \frac{p^3-8p-7-d(p)}{2}, & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

Proof. a) By definition, T_1 is equal to the image of \mathbb{F}_p -points (which are not cusps) on $X(2, 4)_{\mathbb{F}_p}$ under the natural map $g_1 : X(2, 4) \rightarrow X_1(4)$ of degree 2. Since we have $p + 1 - c(2, 4) = 2 \sum_{t \in T_1} 1$ the claim follows.

b) We have $\sum_{t \in T_1} P(t) = \frac{1}{2} \sum_{\substack{t \in g_1(X(2, 4)(\mathbb{F}_p)) \\ t \notin \text{cusps}}} P(t)$. Theorem 4.4 implies

$$\begin{aligned} \text{Tr}(Frob_p|W) &= - \sum_{t \text{ cusp}} \text{Tr}(Frob_p|(i_*\mathcal{F})_t) - \sum_{t \neq \text{cusp}} (p + 1 - P(t)), \\ &= - \sum_{t \text{ cusp}} \text{Tr}(Frob_p|(i_*\mathcal{F})_t) - (p + 1)(p + 1 - c(2, 4)) + \sum_{t \neq \text{cusp}} P(t). \end{aligned}$$

Since $\dim S_3(\Gamma_1(4) \cap \Gamma^0(2)) = 0$, it follows $\text{Tr}(Frob_p|W) = 0$, and the claim follows. Note that we used

$$\sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_*\mathcal{F})_t) = \begin{cases} 4, & \text{if } p \equiv 1 \pmod{4}, \\ 0, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

c) We have $\sum_{t \in T_1} P(t)^2 = \frac{1}{2} \sum_{\substack{t \in g_1(X(2, 4)(\mathbb{F}_p)) \\ t \notin \text{cusps}}} P(t)^2$. Theorem 4.4 implies

$$\begin{aligned} \text{Tr}(Frob_p|W) &= - \sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_*\text{Sym}^2\mathcal{F})_t) - \sum_{t \notin \text{cusps}} ((p + 1 - P(t))^2 - p), \\ &= - \sum_{t \notin \text{cusps}} P(t)^2 + 2(p + 1) \sum_{t \notin \text{cusps}} P(t) - (p + 1 - c(2, 4))(p^2 + p + 1) \\ &\quad - \sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_*\text{Sym}^2\mathcal{F})_t). \end{aligned}$$

It follows from Proposition 4.1d) that $\text{Tr}(Frob_p|W) = d(p)$. The claim follows. Note that we used

$$\sum_{t \in \text{cusps}} \text{Tr}(Frob_p|(i_*\text{Sym}^2\mathcal{F})_t) = 4.$$

□

5.5. $X(4, 8)$. For $t \in \mathbb{F}_p, t \neq 0, 1$, denote by E'_t the elliptic curve $E_t/\langle 2R \rangle$. The curve E'_t is given by the equation $E'_t : y^2 = (x - 2t)(x + 2t)(x - 2t + 4)$, and is isomorphic to the Legendre elliptic curve \mathcal{E}_{1-t} where $\mathcal{E}_t : y^2 = x(x - 1)(x - t)$. Modular curve $X(4, 8)$ is a moduli space for (generalized) elliptic curves with (linearly independent) points of order 8 and 4 with the fixed value of Weil pairing. We have a map $g : X(4, 8) \rightarrow X(2)$, given by the $(E, Q, P) \mapsto (E, 2Q, 4P)$, where Q and P are points on E of order 4 and 8 respectively. The degree of this map is 16 (note that we identify (E, Q, P) with

$(E, -Q, -P)$ and take in account only those pairs (Q, P) which satisfy the Weil pairing condition). Denote by $\tilde{g} : X(2, 8) \mapsto X(2)$ the map given by $(E, Q, P) \mapsto (E, Q, 4P)$.

Proposition 5.5. *a) If $p \equiv 1 \pmod{4}$ and $t \in T_5$, then \mathcal{E}_{1-t} has full \mathbb{F}_p -rational 4-torsion, and the point $(1, 0) \in \mathcal{E}_{1-t}$ is divisible by 4.
b) If p is an odd prime and $t \in T_4$, then the point $(1, 0) \in \mathcal{E}_{1-t}$ is divisible by 4.
c) If $p \equiv 3 \pmod{4}$ then there are no elliptic curves over \mathbb{F}_p with full 4-torsion over \mathbb{F}_p .*

Proof. a) and b) Let $S \in E_t$ be such that $2S \in E_t(\mathbb{F}_p)$, $4S = R$ and $\langle S \rangle$ is \mathbb{F}_p -rational. Then $S + \langle 2R \rangle \in E_t / \langle 2R \rangle$ is \mathbb{F}_p -rational and has order 8. The point $4S + \langle 2R \rangle$ maps to the point $(1, 0) \in \mathcal{E}_{1-t}$ under $E_t / \langle 2R \rangle \cong \mathcal{E}_{1-t}$.

If $p \equiv 1 \pmod{4}$, and $T \in E_t(\mathbb{F}_p)$ of order 2, $T \neq 2R$, we have by Proposition 3.1a) and c), that $x(T)$ is square in \mathbb{F}_p and that $P^\sigma - P \in \{\mathcal{O}, 2R\}$, for all $\sigma \in \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$, where $2P = T$. It follows that $P + \langle 2R \rangle$ is \mathbb{F}_p -rational of order 4.

c) If $y^2 = (x - a)(x - b)(x - c)$ is an elliptic curve over \mathbb{F}_p with $a, b, c \in \mathbb{F}_p$, then it follows from the descent homomorphism that, for example, the point $(a, 0)$ is divisible by 2 over \mathbb{F}_p if and only if $a - b$ and $a - c$ are squares in \mathbb{F}_p . If both $(a, 0)$ and $(b, 0)$ are divisible by 2 (i.e. if the elliptic curve has full \mathbb{F}_p -rational 4-torsion), then both $a - b$ and $b - a$ are squares, hence -1 is a square in \mathbb{F}_p , and $p \equiv 1 \pmod{4}$. \square

Proposition 5.6. *a)*

$$\sum_{t \in T_5} 1 = \begin{cases} \frac{p-a(p)-15}{16}, & \text{if } p \equiv 1 \pmod{8}, \\ \frac{p-3}{8}, & \text{if } p \equiv 3 \pmod{8}, \\ \frac{p-a(p)-7}{16}, & \text{if } p \equiv 5 \pmod{8}, \\ \frac{p-7}{8}, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

b)

$$\sum_{t \in T_4} 1 = \begin{cases} \frac{3p+a(p)-21}{16}, & \text{if } p \equiv 1 \pmod{8}, \\ \frac{p-3}{4}, & \text{if } p \equiv 3 \pmod{8}, \\ \frac{3p+a(p)-13}{16}, & \text{if } p \equiv 5 \pmod{8}, \\ \frac{p-7}{4}, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Proof. a) Let $p \equiv 1 \pmod{4}$. It follows from Proposition 3.1 that $\#T_5$ is equal to the number of t 's for which the points $(1, 0), (0, 0) \in \mathcal{E}_t$ are divisible by 4 and 2 (in $\mathcal{E}_t(\mathbb{F}_p)$) respectively, which in turn, by Proposition 3.1 is equal to the number of the point in the image of \mathbb{F}_p -point of $X(4, 8)$ under the map g , i.e. $\#T_4 = \#g(X(4, 8)(\mathbb{F}_p))$.

Note that $f_1(\tau)$ is the modular form that corresponds under the modularity theorem to the elliptic curve $X(4, 8)$, hence $\#X(4, 8)(\mathbb{F}_p) = p + 1 - a(p)$. Also, if one point in the preimage of g is \mathbb{F}_p -rational, then the same holds for all the points in the preimage, so we have that $p + 1 - a(p) - c(4, 8) = 16 \sum_{t \in T_5} 1$, and the claim follows.

If $p \equiv 3 \pmod{4}$ then $\#T_5 = \#T_2$ (by Proposition 3.1 a)), and the claim follows from

Proposition 5.2.

b) Similar to the part a), it follows from Proposition 3.1 that $\#T_4$ is equal to the number of elements in the image of \mathbb{F}_p -rational points of $X(2, 8)$ under the map \tilde{g} .

Let $p \equiv 1 \pmod{4}$. There are $\#T_5$ points in the image of \tilde{g} which are also in the image of g , hence each of these points have eight \mathbb{F}_p -rational points in the preimage by the map \tilde{g} , while the remaining $\#T_4 - \#T_5$ points have four \mathbb{F}_p -rational points in the preimage. Hence $8\#T_5 + 4\#(T_4 - \#T_5) = p + 1 - c(2, 8)$, and the claim follows.

Let $p \equiv 1 \pmod{4}$. There are $\#T_5$ elements in the image of \tilde{g} which are also in the image of g , hence each of these elements have eight \mathbb{F}_p -rational points in the preimage by the map \tilde{g} , while the remaining $\#T_4 - \#T_5$ elements have four \mathbb{F}_p -rational elements in the preimage. Hence $8\#T_5 + 4\#(T_4 - \#T_5) = p + 1 - c(2, 8)$, and the claim follows.

If $p \equiv 3 \pmod{4}$ then by Proposition 3.1 there are no elliptic curves over \mathbb{F}_p with full 4-torsion over \mathbb{F}_p , hence $4\#T_4 = p + 1 - c(2, 8)$, and the claim follows. \square

Next we prove that if $p \equiv 3 \pmod{4}$ then $\#T_0 = \frac{1}{2}\#T_1$ and $\sum_{t \in T_0} P(t) = \frac{1}{2} \sum_{t \in T_1} P(t)$. By definition, $t \in T_1$ implies that $1 - t = u^2$ for some $u \in \mathbb{F}_p$. Denote by $t' = 1 - \left(\frac{1}{u}\right)^2$. It follows that $t' \in T_1$ and $(t')' = t$. Moreover, only one of $t = 1 - u^2$ and $t' = \frac{u^2 - 1}{u^2}$ is a square, hence precisely one of them is an element of T_0 . It follows that $\#T_0 = \frac{1}{2}\#T_1$. The second equality now follows from the fact that $P(t) = P(t')$ (it is easy to check that E_t and $E_{t'}$ have the same j -invariants).

Theorem 1.1 now follows from Proposition 3.2, Proposition 3.3, Propositions 5.1-5.6 and the previous discussion.

6. DIOPHANTINE m -TUPLES IN \mathbb{F}_p AND CHARACTER SUMS

In this section, we will use properties of character sums (sums of the Legendre symbols) to show that for arbitrary $m \geq 2$ there exist Diophantine m -tuples in \mathbb{F}_p for sufficiently large p . We will also derive formulas for the number of Diophantine pairs and triples in \mathbb{F}_p .

Theorem 6.1. *Let $m \geq 2$ be an integer. If $p > 2^{2m-2}m^2$, then there exist a Diophantine m -tuple in \mathbb{F}_p .*

Proof. We prove the theorem by induction on m . For $m \geq 2$ and $p > 16$ (in fact, for $p \geq 5$), we may take the Diophantine pair $\{1, 3\}$ in \mathbb{F}_p .

Assume that the statement holds for an integer $m \geq 2$. Take a prime $p > 2^{2m}(m+1)^2$. Since $p > 2^{2m-2}m^2$, there exist a Diophantine m -tuple $\{a_1, \dots, a_m\}$ in \mathbb{F}_p . Let

$$g := \#\{x \in \mathbb{F}_p : \left(\frac{a_i x + 1}{p}\right) = 1, \text{ for } i = 1, \dots, m\}.$$

and denote by \bar{a}_i the multiplicative inverse of a_i in \mathbb{F}_p . Then, by [10, Exercise 5.64], we have

$$\begin{aligned} g &= \#\{x \in \mathbb{F}_p : \left(\frac{x + \bar{a}_i}{p}\right) = \left(\frac{\bar{a}_i}{p}\right), \text{ for } i = 1, \dots, n\} \\ &\geq \frac{p}{2^m} - \left(\frac{m-2}{2} + \frac{1}{2^m}\right) \sqrt{p} - \frac{m}{2}. \end{aligned}$$

Since,

$$\left(\frac{m-2}{2} + \frac{1}{2^m}\right) \sqrt{p} + \frac{m}{2} + (m+1) \leq \sqrt{p} \left(\frac{m}{2} - 1 + \frac{1}{2^m} + \frac{3}{2^{m+1}}\right) < \frac{m}{2} \sqrt{p} < \frac{p}{2^m},$$

we get that $g > m+1$. Thus, we conclude that there exist $x \in \mathbb{F}_p$, $x \notin \{0, a_1, a_2, \dots, a_m\}$, such that $\left(\frac{a_i x + 1}{p}\right) = 1$ for $i = 1, \dots, m$. Hence, $\{a_1, \dots, a_m, x\}$ is a Diophantine $(m+1)$ -tuple in \mathbb{F}_p . \square

In the proof of next two propositions we will several times use the following well-know fact (see e.g. [7, Section 7.8]):

$$\sum_{x \in \mathbb{F}_p} \left(\frac{\alpha x^2 + \beta x + \gamma}{p}\right) = -\left(\frac{\alpha}{p}\right),$$

provided $\beta^2 - 4\alpha\gamma \not\equiv 0 \pmod{p}$.

Proposition 6.2. *Let p be an odd prime. The number of Diophantine pairs in \mathbb{F}_p is equal to*

$$N^{(2)}(p) = \begin{cases} \frac{(p-1)(p-2)}{4}, & \text{if } p \equiv 1 \pmod{4}, \\ \frac{p^2-3p+4}{4}, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. We have

$$4N^{(2)}(p) = \sum_{a, b \neq 0, a \neq b} \left(1 + \left(\frac{ab+1}{p}\right)'\right),$$

where $\left(\frac{x}{p}\right)' = \left(\frac{x}{p}\right)$ for $x \neq 0$ and $\left(\frac{0}{p}\right)' = 1$. Therefore, we have

$$\begin{aligned} 4N^{(2)}(p) &= \sum_{b \neq 0} \sum_{a \neq 0, b} 1 + \sum_{b \neq 0} \sum_{a \neq 0, b} \left(\frac{ab+1}{p}\right) + \sum_{b \neq 0, b^2 \neq -1} 1 \\ &= (p-1)(p-2) + \sum_{b \neq 0} \left(-1 - \left(\frac{b^2+1}{p}\right)\right) + \sum_{b \neq 0, b^2 \neq -1} 1. \end{aligned}$$

If $p \equiv 1 \pmod{4}$, the last sum is equal to $p-3$. Thus we get

$$4N^{(2)}(p) = (p-1)(p-2) - (p-1) + 2 + (p-3) = (p-1)(p-2).$$

Similarly, for $p \equiv 3 \pmod{4}$, we get

$$4N^{(2)}(p) = (p-1)(p-2) - (p-1) + 2 + (p-1) = p^2 - 3p + 4.$$

□

Proposition 6.3. *Let p be an odd prime. The number of Diophantine triples in \mathbb{F}_p is equal to*

$$N^{(3)}(p) = \begin{cases} \frac{(p-1)(p-3)(p-5)}{48}, & \text{if } p \equiv 1 \pmod{4}, \\ \frac{(p-3)(p^2-6p+17)}{48}, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. We have

$$48N^{(3)}(p) = \sum_S \left(1 + \left(\frac{ab+1}{p} \right)' \right) \left(1 + \left(\frac{ac+1}{p} \right)' \right) \left(1 + \left(\frac{bc+1}{p} \right)' \right),$$

where the sum is taken over all triples a, b, c in \mathbb{F}_p such that $a, b, c \neq 0$ and $a \neq b \neq c \neq a$. Let us denote:

$$\begin{aligned} S_1 &= \sum_S 1, \\ S_2 &= \sum_S \left(\frac{ab+1}{p} \right), \\ S_3 &= \sum_S \left(\frac{ab+1}{p} \right) \left(\frac{ac+1}{p} \right), \\ S_4 &= \sum_S \left(\frac{ab+1}{p} \right) \left(\frac{ac+1}{p} \right) \left(\frac{bc+1}{p} \right), \\ S_5 &= \sum_{S'} 1, \\ S_6 &= \sum_{S'} \left(\frac{ab+1}{p} \right), \\ S_7 &= \sum_{S'} \left(\frac{ab+1}{p} \right) \left(\frac{-a^{-1}b+1}{p} \right), \end{aligned}$$

where the sums S_5, S_6, S_7 are taken over all pairs a, b in \mathbb{F}_p such that $a, b \neq 0$, $b \neq a, -a^{-1}$, $a^2 \neq -1$. Then we have

$$(6.1) \quad 48N^{(3)}(p) = S_1 + 3S_2 + 3S_3 + S_4 + 3S_5 + 6S_6 + 3S_7.$$

Thus, it remains to compute the sums S_1, \dots, S_7 . We will derive the formulas for the cases $p \equiv 1, 3 \pmod{4}$. Where the formulas for these two cases differ, the upper sign will correspond to $p \equiv 1 \pmod{4}$, while the lower sign will correspond to $p \equiv 3 \pmod{4}$.

We have

$$S_1 = (p-1)(p-2)(p-3),$$

$$S_2 = (p-3) \sum_{a,b \neq 0, a \neq b} \left(\frac{ab+1}{p} \right) = -(p-3)^2,$$

$$\begin{aligned} S_3 &= \sum_{a,b \neq 0, a \neq b} \left(\frac{ab+1}{p} \right) \sum_{c \neq 0, a, b} \left(\frac{ac+1}{p} \right) \\ &= \sum_{a,b \neq 0, a \neq b} \left(-1 - \left(\frac{ab+1}{p} \right) - \left(\frac{a^2+1}{p} \right) \right) \\ &= (p-3) - \sum_{a,b \neq 0, a \neq b, ab+1 \neq 0} 1 - \sum_{a,b \neq 0, a \neq b} \left(\frac{ab+1}{p} \right) \left(\frac{a^2+1}{p} \right) \\ &= (p-3) - (p^2 - 4p + 4 \pm 1) - \sum_{a \neq 0} \left(\frac{a^2+1}{p} \right) \left(-1 - \left(\frac{a^2+1}{p} \right) \right) \\ &= (p-3) - (p^2 - 4p + 4 \pm 1) - 2 + (p-2 \mp 1) = -p^2 + 6p - 11 \mp 2, \end{aligned}$$

$$\begin{aligned} S_4 &= \sum_{a,b \neq 0, a \neq b} \left(\frac{ab+1}{p} \right) \sum_{c \neq 0, a, b} \left(\frac{ac+1}{p} \right) \left(\frac{bc+1}{p} \right) \\ &= \sum_{a,b \neq 0, a \neq b} \left(\frac{ab+1}{p} \right) \left(- \left(\frac{ab}{p} \right) - 1 - \left(\frac{a^2+1}{p} \right) \left(\frac{ab+1}{p} \right) - \left(\frac{b^2+1}{p} \right) \left(\frac{ab+1}{p} \right) \right) \\ &= - \sum_{a \neq 0} \sum_{t \neq 0, a^2} \left(\frac{t+1}{p} \right) \left(\frac{t}{p} \right) + \sum_{a \neq 0} \left(1 + \left(\frac{a^2+1}{p} \right) \right) - 2 \sum_{a,b \neq 0, a \neq b, ab+1 \neq 0} \left(\frac{a^2+1}{p} \right) \\ &= - \sum_{a \neq 0} \left(-1 - \left(\frac{a^2+1}{p} \right) \right) + \sum_{a \neq 0} \left(1 + \left(\frac{a^2+1}{p} \right) \right) \\ &\quad - 2 \sum_{a \neq 0} \left(-1 - 1 - \left(\frac{a^2+1}{p} \right) - \left(\frac{a^{-2}+1}{p} \right) \right) \\ &= (4p-12) + 2((p-1)-2) = 8p-18, \end{aligned}$$

$$S_5 = p^2 - 5p + 6 \mp (p-3),$$

$$S_6 = \sum_{a \neq 0, a^2 \neq -1} \left(-1 - \left(\frac{a^2+1}{p} \right) \right) = -p + 4 \pm 1,$$

$$\begin{aligned}
 S_7 &= \sum_{a \neq 0, a^2 \neq -1} \sum_{c \neq 0, a, -a^{-1}} \left(\frac{ac+1}{p} \right) \left(\frac{-a^{-1}c+1}{p} \right) \\
 &= \sum_{a \neq 0, a^2 \neq -1} (\mp 1 - 1) = -p + 3 \mp (p-3).
 \end{aligned}$$

Putting all these formulas together in (6.1), we get

$$48N^{(3)}(p) = p^3 - 9p^2 + 29p - 33 \mp (6p - 18),$$

and by writing separately the cases $p \equiv 1, 3 \pmod{4}$, we obtain the formula for $N^{(3)}(p)$ given in the statement of the proposition. \square

For small values for m , the bound from Theorem 6.1 can be improved by direct calculation, or by applying Propositions 6.2 and 6.3 and Theorem 1.1. We get that $N^{(2)}(p) > 0$ for $p \geq 3$, $N^{(3)}(p) > 0$ for $p \geq 7$, $N^{(4)}(p) > 0$ for $p \geq 11$, $N^{(5)}(p) > 0$ for $p \geq 23$.

We can follow the proof of Proposition 6.3 to sketch the proof of the asymptotic formula $N^{(m)}(p) = \frac{1}{2^{\binom{m}{2}}} \frac{p^m}{m!} + o(p^m)$. Indeed, we have

$$m! 2^{\binom{m}{2}} N^{(m)}(p) = \sum \prod_{1 \leq i < j \leq m} \left(1 + \left(\frac{a_i a_j + 1}{p} \right)' \right),$$

where the sum is taken over all m -tuples a_1, \dots, a_m of distinct non-zero elements of \mathbb{F}_p . The main term comes from $\sum 1 = (p-1)(p-2)\dots(p-m) = p^m + o(p^m)$, while all other terms are of the form

$$\sum_{a_1, \dots, a_{m-1}} \sum_{a_m} \left(\frac{f(a_m)}{p} \right),$$

where $f(x)$ is a non-square polynomial of degree $\leq m-1$ and the sums are taken over almost all m -tuples in \mathbb{F}_p . By Weil's estimate for character sums (see e.g. [10, Theorem 5.41]), we conclude that the contribution of all these terms is $O(p^{m-1} \sqrt{p}) = o(p^m)$.

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